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The effect of a free surface on the normal modes of a stratified shear flow

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Summary

The effect of the presence of a free surface on the behaviour of shear-modified internal waves is investigated. In all cases, the qualitative behaviour with varying Richardson number is similar to that in the case of a rigid upper boundary. The quantitative differences are shown to be small and therefore the free surface can be replaced by a rigid-lid boundary condition with negligible error.

1. Introduction

Internal gravity waves, the normal modes of a stratified fluid, have been studied extensively over the years and their importance recognised in such fields as oceanography and meteorology. In many application, however, there is an underlying shear flow which may have a substantial effect on the behaviour of the wave modes. Although such shear has been considered in problems such as the generation of lee waves, it was only fairly recently that the effect of shear on the free oscillations (the normal modes) was investigated.

Banks, Drazin and Zaturska [1] conducted a systematic investigation of the class of discrete, non-singular stable modes of an inviscid, stratified plane-parallel shear flow; these are shear-modified internal waves. They examined the asymptotic behaviour of the modes for both large and small Richardson numbers, both analytically and numerically. Their most important finding was that for small Richardson numbers, the wave speed depended critically on the local behaviour of the mean flow near the maximum (or minimum) of the velocity profile. Their results were applicable to flows between two rigid horizontal boundaries and to unbounded flows satisfying various constraints on the buoyancy frequency as $z \to \infty$.

Here, the effect on the behaviour of shear-modified internal wave modes of a free upper surface is considered. In Sec. 2, the Taylor-Goldstein equation governing the motion and the appropriate form of the free-surface condition are briefly derived. The following sections deal respectively with the analytic asymptotic results for large and small Richardson number and with some numerical results. The former indicate that the presence of the

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free surface makes no real qualitative difference to the results of Banks et al. [1]; the numerical results show that the quantitative differences between the two cases are also small, and give some indication of the error arising from replacing a free surface by a rigid lid.

2. The governing equations

We consider disturbances in a two-dimensional incompressible stratified fluid, whose basic state is one of plane parallel flow, i.e.

$$\boldsymbol{u}^* = \bar{u}^*(z^*)\boldsymbol{i}, \quad \rho^* = \bar{\rho}^*(z), \quad p^* = \bar{p}^*(z),$$

with

$$\frac{\mathrm{d}\bar{p}^*}{\mathrm{d}z^*} = -g\bar{\rho}^*(z^*).$$

(* denotes dimensional variables). Furthermore, we make the Boussinesq approximation, neglecting density variation with depth in all but the buoyancy terms. Non-dimensionalization can be carried out with respect to characteristic values of velocity, density and length scale; these are chosen to be, respectively, the maximum flow speed of the basic flow, the Boussinesq reference density ρ_0 (taken to be the free-surface value) and the channel depth L, which gives a scale for the basic shear flow.

The non-dimensional Boussinesq equation are then

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -G(\nabla p + \rho \boldsymbol{k}),$$

where $G = gL/U^2$ is an inverse Froude number. As the disturbances are regarded as small perturbations to the basic flow, we write

$$u = \bar{u}(z) + u', \quad w = w', \quad \rho = \bar{\rho}(z) + \rho',$$

the primes denoting the perturbation quantities. Substitution of these, followed by linearization and the introduction of a stream function ψ (such that $u' = \psi_z$, $w' = -\psi_x$), leads to the following equations for ψ and the density perturbation (where we now drop the primes):

$$\rho_t + \bar{u}\rho_x - \psi_x\bar{\rho}_z = 0,$$

$$\psi_{zzt} + \bar{u}\psi_{xzz} - \bar{u}_{zz}\psi_x + \psi_{xxt} + \bar{u}\psi_{xxx} - G\rho_x = 0.$$

If normal modes of the form

$$\psi = \phi(z)\exp\{i\alpha(x-ct)\}, \quad \rho = \hat{\rho}(z)\exp\{i\alpha(x-ct)\}$$

are sought, the above equations can be combined to yield the Taylor-Goldstein equation

$$(\bar{u}-c)(\phi''-\alpha^2\phi) - u'\phi + \frac{JN^2\phi}{\bar{u}-c} = 0. \tag{2.1}$$

Here, J is an overall Richardson number and

$$JN^2 = -\frac{G}{\rho}\frac{\partial\rho}{\partial z} = -\frac{gL^2}{U^2}\frac{1}{\rho^*}\frac{\partial\rho^*}{\partial z^*}. \tag{2.2}$$

In a similar fashion, the bottom boundary condition is found to be

$$\phi = 0 \quad \text{on} \quad z = -1; \tag{2.3}$$

the free-surface condition reduces to

$$(\bar{u}-c)^2\phi' - \bar{u}'(\bar{u}-c)\phi - G\phi = 0 \quad \text{on} \quad z = 0. \tag{2.4}$$

3. Asymptotic behaviour for large and small J

In their study of the corresponding problem with a rigid upper boundary, Banks et al. [1] considered the asymptotic behaviour of the shear-modified internal waves in the limits of large and small overall Richardson number. Similar asymptotic techniques can be used here. In fact, the results differ only in detail from the findings of Banks et al.; accordingly, only a summary of the most important features is given.

For the sake of simplicity and since the primary interest lies in the effect of the nature of the shear flow, the buoyancy frequency will be assumed to be constant with height throughout the following analysis. Thus, in the notation of Sec. 2, $N^2 \equiv 1$, without loss of generality.

(a) Large J behaviour

The limit $J \to \infty$ corresponds to either infinite buoyancy or zero dimensional scale of the basic shear flow; shear effects are therefore expected to be negligible compared to that of the stratification. In this limit, the solutions should tend to the normal spectrum of internal wave modes. (We will neglect here the additional free-surface modes.)

That this is the case can be verified by expanding the Taylor-Goldstein equation, and the boundary conditions, in powers of $J^{-1/2}$, as in Banks et al. [1]. For J large, the lowest-order eigenfunctions are

$$\phi_n = \sin\beta_n(z+1) \tag{3.1}$$

with associated eigenvalues (the phase speeds)

$$c_n = \pm\left(\frac{J}{\beta_n^2 + \alpha^2}\right)^{1/2}, \tag{3.2}$$

where the β_n $(n = 1, 2, \ldots)$ are solutions of

$$\left(\frac{\beta_n^2 + \alpha^2}{\beta_n}\right)\tan\beta_n = \frac{J}{G}. \tag{3.3}$$

The n^{th} root β_n lies in the interval $(n\pi, n\pi + \pi/2)$.

Although we are here concerned with J large, it is the ratio J/G which is important. This is effectively the ratio of the length scale of the basic flow to that of the density variations, and in the context in which we are interested (oceanic applications), this will usually be small, even for fairly substantial values of J. (This is also in keeping with our use of the Boussinesq approximation.) In this case, examination of the transcendental equation for β_n indicates that

$$\beta_n = n\pi + \epsilon_n,$$

where ϵ_n is small. Therefore, the error in approximating β_n by $n\pi$ in the wave speed should be small also. In other words, the rigid-lid wave speed is a good approximation to the wave speed of the internal modes in the presence of a free surface. Some numerical checks of this are presented in Sec. 4.

Higher-order corrections to the wave speed and to the vertical structure $\phi(z)$, due to the shear, can be calculated. Any such terms are, however, dependent upon the precise structure of the basic flow and there is not much to be gained by considering in detail any one such example. It is, however, perhaps worth noting that the presence of the free surface makes the problem asymmetric, so that even for a $\bar{u}(z)$ symmetric about the mid-depth of the channel there will always be an $O(1)$ correction to the wave speed c.

(b) Small Richardson number

When examining the normal modes of shear flows between rigid boundaries at low Richardson number, Banks et al. found, first from their numerical results and later by asymptotic analysis, that the nature, position and strength of the maximum speed of the basic flow exerted a controlling influence on the wave speed of the disturbances. Two cases emerged, one when the maximum lay in the flow interior, the other when it was attained at a boundary.

In the first case, as the Richardson number J decreased to zero, the wave speed of all the normal modes decreased to the maximum speed of the basic flow; for small J, the decay of all modes and for all wavelengths, was algebraic. The asymptotic analysis revealed that

$$c \sim 1 - \frac{2JN_m^2}{\bar{u}_m'' n(n+2)}, \quad \text{as} \quad J \to 0, \qquad \text{for} \quad n = 1, 2, \ldots, \tag{3.4}$$

where the subscript m denotes the value at the position of the basic flow maximum and n is the mode number index. This result is clearly dependent only on the local character of the flow in the region of its maximum speed. Intuitively, then, we would expect the replacement of a rigid upper boundary by a free surface to have no real effect on this limiting form. That this is so can be confirmed by repeating the asymptotic analysis of Banks et al. with the new boundary condition. Equation (3.4) is thus the appropriate asymptotic form for the free-surface problem also, if the basic flow maximum is in the flow interior. The precise nature of the eigenfunctions, in the "outer" solution, will be affected by the boundary conditions, but this is not of great significance.

For flows where the maximum speed is attained at a boundary (with non-zero shear there), Banks et al. again found that $\bar{u}_{\max}$ was the limiting value of the wave speed for all modes. However, the nature of the approach to the limit showed two distinct patterns, and the limit was not attained as $J \downarrow 0$. A finite number of modes tended to the limit in an algebraic manner, as J decreased to some value J_n (depending upon the mode). The remainder showed exponential decay as J decreased to $(\bar{u}_m)^2/4N_m^2$ (subscript m denoting the values at the basic flow maximum).

When the maximum is found at the free surface, we might well expect the different boundary condition there to make some difference to the results from the rigid lid case. The analysis of Banks et al. can be applied to such a flow, where in our notation,

$$\bar{u}(z) < \bar{u}(0) = \bar{u}_m \equiv 1, \quad -1 \leqslant z < 0,$$

and

$$\bar{u}'(0) = \bar{u}'_m > 0.$$

The free-surface condition does affect the inner solution ϕ_i, which becomes

$$\phi_i = \frac{1-c}{\bar{u}'_m}\frac{1}{2\nu d}(Z+1)^{1/2}\{(\nu - \tfrac{1}{2} + d)(Z+1)^{\nu} + (\nu + \tfrac{1}{2} - d)(Z+1)^{-\nu}\} \tag{3.5}$$

where

$$\nu = \left(\frac{1}{4} - \frac{JN_m^2}{(\bar{u}'_m)^2}\right)^{1/2} \tag{3.6}$$

and

$$d = 1 + \frac{G}{\bar{u}'_m(1-c)};$$

this should be compared to Eqn. (65) of Banks et al. However, if the matching procedure is carried through, the same two types of behaviour are found as in the rigid-lid problem, according to whether ν is real or imaginary.

For ν real, a finite number (possibly zero) of modes decay with

$$c \sim 1 + \bar{u}'_m \left\{ D_1 (J - J_n) \frac{\nu_n + d - \frac{1}{2}}{\nu_n - d + \frac{1}{2}} \right\}^{1/2\nu_n} \tag{3.7}$$

as J decreases to some critical value J_n. (D_1 is a constant; the subscript n denotes the dependence on the mode number.) For ν purely imaginary, i.e. for

$$\frac{J N_m^2}{(\bar{u}'_m)^2} < \frac{1}{4},$$

the modes decay exponentially, with

$$c \sim 1 + \bar{u}'_m \exp(-C'/D') \exp(-j\pi/\mu), \quad j = 1, 2, \ldots, \tag{3.8}$$

where $\mu = \mathrm{i}\nu$ and C', D' are constants, as $\mu \downarrow 0$.

The equations (3.7) and (3.8) show (by comparison with Eqns. (67) and (69) of Banks et al.) that there is no real qualitative difference in behaviour between the rigid-lid and free-surface cases for flows with maximum at the upper surface, although quantitative differences do exist. These are explored more fully in the section on numerical results.

Thus, we can conclude that wherever the basic flow has its maximum speed, the substitution of a free surface for a rigid lid will have no real qualitative effect at small Richardson number.

4. Some numerical results

Although the asymptotic results presented in the previous section indicate that the replacement of a free upper surface by a rigid boundary should have little effect on the shear-modified internal waves, it is useful to have some numerical results for comparison over a large range of values of J, and not just for the extremes of very large and small J. These can also give some quantitative measure of the error in using a rigid-lid approximation, for typical parameter values.

Use was made in the calculations of several routines from the widely-available NAG library of numerical routines. These were all essentially similar, being designed to solve eigenvalues problems of the Sturm-Liouville type, namely of the form

$$(p(z)f')' + q(z; \lambda)f = 0,$$

where λ is an eigenvalue and f its associated eigenfunction. The Sturm-Liouville equation is transformed by a scaled Prüfer transform to the (pf', f) phase plane. The Prüfer variables B, ϕ and ρ are defined by

$$p(z)f' = B^{1/2} \exp\frac{\rho}{2} \cos\frac{\rho}{2},$$

$$f = B^{1/2} \exp\frac{\rho}{2} \sin\frac{\phi}{2}.$$

(For a general discussion of Prüfer transforms, see Birkhoff and Rota [2,Chap. X]; details of the scaled Prüfer transform used in the NAG routines are given by Pryce [3].) In the transformed problem, it is easy to identify and select the n^{th} eigenvalue, by the number of zeros of the eigenfunction, and there is a unique correspondence between solutions of the Prüfer system and the original differential equation. The problem in the phase plane is solved using a shooting method, the iterations being carried out until the required degree of accuracy is achieved.

The Taylor-Goldstein equation

$$(\bar{u}-c)(\phi''-\alpha^2\phi)-\bar{u}''\phi+\frac{JN^2\phi}{\bar{u}-c}=0$$

can be written in the appropriate S-L form for use with these routines, with $p(z)=1$ and

$$q(z,c)=-\alpha^2-\frac{\bar{u}''}{\bar{u}-c}+\frac{JN^2}{(\bar{u}-c)^2}$$

where c is the eigenvalue to be determined for fixed J, $N^2(z)$ and α. For the rigid-lid problem, where both top and bottom boundary conditions take the form $\phi=0$, the simplest routine (D02KAF) can be used. However, in the free-surface problem, the boundary condition at the surface,

$$(\bar{u}-c)^2\phi'-\bar{u}'(\bar{u}-c)\phi-G\phi=0 \quad \text{on} \quad z=0,$$

clearly involves the unknown eigenvalue c. Use of the more complex routine D02KDF allows the latest approximation to the eigenvalue to be substituted into the boundary condition at each iteration.

The density variation in the examples considered was taken to be constant, so that $N^2=1$, and J alone was varied; in addition, all the basic flows were confined to the region $-1\leqslant z\leqslant 0$. Calculations were carried out over a large range of values of J. For the first and largest, the initial estimate for the eigenvalue supplied to the routine was based on the asymptotic limit value of c as $J\to\infty$ (see Sec. 3). At subsequent values of J, the initial guess for c was based on the final result for the previous calculation or on a linear extrapolation of the previous two results.

The remaining parameters in the problem are α, the wave number, G the inverse Froude number and k, the eigenvalue (mode) index. Calculations were carried out at a number of values of α for the first three internal-wave modes ($k=1, 2, 3$; $k=0$ corresponds to a surface-wave mode). Similarly, a number of values of G were considered. In terms of the length and velocity scales L and U, $G=gL/U^2$; for an oceanic flow, U is typically of the order of 10 cm/sec. and length scales of the order of kilometres, giving $G\sim O(10^6)$. The calculations were therefore performed systematically for $G=10^5$, 10^6 and 10^7; a few isolated calculations were also made with much larger G.

Two velocity profiles were considered, one with a maximum in the flow interior, the other with the maximum at the upper boundary. The first of these was the sinusoidal profile

$$\bar{u}(z)=\sin(2\pi z+\pi), \quad -1\leqslant z\leqslant 0.$$

By considering a particular eigenmode, for a fixed wavelength, the effect of varying the

Table 1. Wave speeds of the lowest internal-wave mode, with wave number $\alpha = 0$, for a range of values of G and J. The corresponding rigid-lid results are shown for comparison.

J	G				
	10^5	10^6	10^7	10^{10}	Rigid lid
400	6.3744	6.3761	6.3763	6.3763	6.3662
300	5.5239	5.5250	5.5251		5.5133
200	4.5158	4.5163	4.5164		4.5016
100	3.2053	3.2055	3.2055		3.1831
40	2.0565	2.0565	2.0565		2.0565
20	1.5098	1.5098	1.5098		1.5098

inverse Froude number G can be evaluated, over a large range of values of J, the Richardson number. Some typical results are shown in Table 1. From such results, it is clear that G does not make much difference to the wave speed c; in fact, the error in replacing the free-surface condition by a rigid-lid condition is of the order of a few percent at most.

Figures 1 and 2 show the wave speed c as a function of Richardson number J, at fixed Froude number, for the first three wave modes of given wavelength and for the lowest mode at a number of values of wave number α, respectively. From these, and similar results, it is clear that c decreases to 1 (i.e. in dimensional terms, c decreases to $\bar{u}_{\max}$) as J tends to zero. Table 2 gives a comparison, for small J, of the numerically computed values and those obtained from the asymptotic result of Sec. 3, for the first three modes. Again, this is a typical set of results and the asymptotic behaviour for small J is clearly well-modelled by this formula. It should be noted that with this choice of $\bar{u}(z)$, $\mathrm{d}q/\mathrm{d}\lambda$ in the Sturm-Liouville problem may not always be single signed. Some failures of the NAG

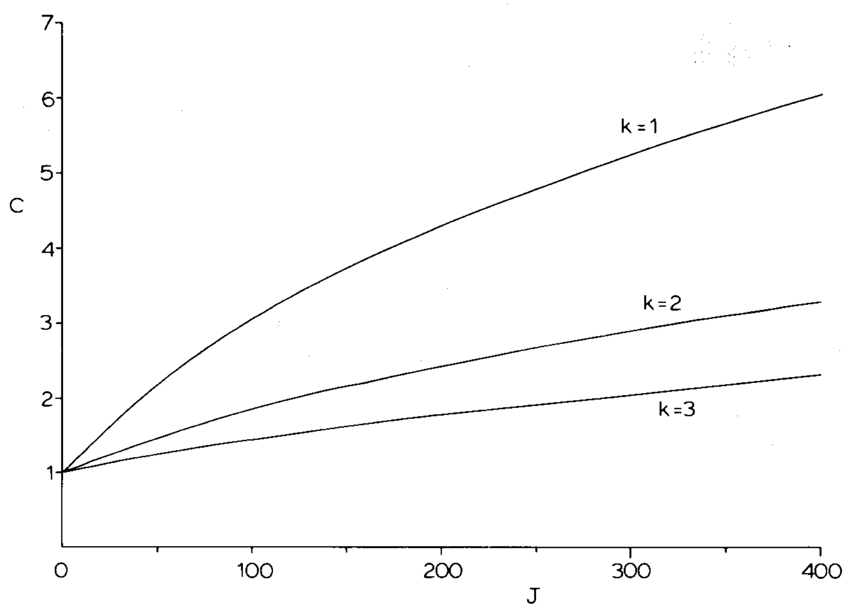

Figure 1. The wave speed c for the first three modified internal-wave modes, in the shear flow $\bar{u}(z) = \sin(2\pi z + \pi)$, for $\alpha = 1$ and $G = 10^5$.

Table 2. Wave speed c for the first three internal-wave modes, at low Richardson number, with $\alpha = 1$, $G = 10^5$; (a) asymptotic results; (b) numerical results.

j	$k = 1$		$k = 2$		$k = 3$	
	(a)	(b)	(a)	(b)	(a)	(b)
0.2	1.00338	1.00338	1.00127	1.00127	1.00068	1.00068
0.4	1.00675	1.00679	1.00253	1.00354	1.00135	1.00135
0.6	1.01013	1.01021	1.00380	1.00382	1.00203	1.00203
0.8	1.01351	1.01365	1.00507	1.00510	1.00270	1.00271
1.0	1.01689	1.01711	1.00633	1.00639	1.00338	1.00340
1.2	1.02026	1.02058	1.00760	1.00768	1.00405	1.00409
1.4	1.02364	1.02408	1.00887	1.00898	1.00473	1.00478
1.6	1.02702	1.02759	1.01032	1.01028	1.00540	1.00547
1.8	1.03039	1.03112	1.01140	1.01160	1.00608	1.00616
2.0	1.03377	1.03467	1.01267	1.01290	1.00676	1.00685
~	~	~	~	~	~	~
4.0	1.06755	1.07136	1.02533	1.02635	1.01351	1.01394

routine for the rigid-lid problem at small values of J may be attributable to this, although no such problems were encountered for the corresponding free-surface problem. Failure is not a necessary consequence if $dq/d\lambda$ is not single-valued, but in general computation time is increased.

The second choice for the basic flow was

$$\bar{u}(z) = z, \quad -1 \leqslant z \leqslant 0;$$

this flow attains its maximum speed at the free-surface level. Again, the variation with G,

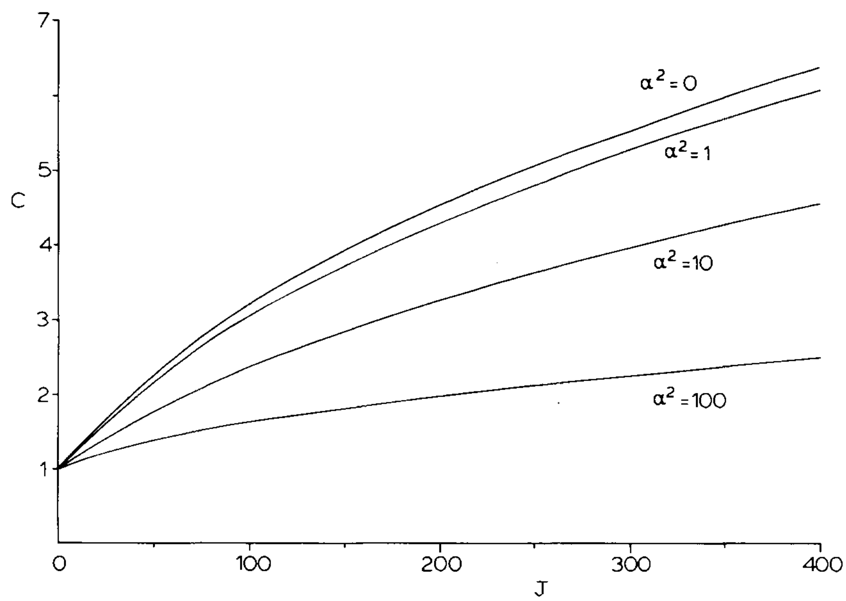

Figure 2. The wave speed c for the lowest mode in the sinusoidal shear flow, for $G = 10^5$ and wave number α, $\alpha^2 = 0, 1, 10$ and 100.

Table 3. A comparison of the wavespeed c for the lowest-mode internal wave, of wave number $\alpha = 1$, for various values of G; the basic flow is the linear shear flow $\bar{u}(z) = z$.

J	G				
	10^5	10^6	10^7	10^{10}	rigid lid
400	5.5764	5.5781	5.5783	5.5783	5.5783
300	4.7662	4.7673	4.7674		4.7674
200	3.8059	3.8064	3.8065		3.8065
100	2.5569	2.5571	2.5571		2.5571
20	0.90992	0.90994	0.90994		0.90994

for values around those applicable in oceanic applications for example, appears to be small and the rigid-lid model a good approximation. Table 3 shows a typical comparison for a given mode and wave number.

Figures 3 and 4 show the variation with mode number and wave number respectively, over a large range of values of J. These, and similar results, again show the decay of the wave speed c to $\bar{u}_{max}$ (here $\bar{u}_{max} = 0$ at $z = 0$). The scale of these figures, however, masks the exponential character of the variation of c at $J \downarrow \frac{1}{4}$. Close examination (Figure 5) for small J does indicate that the decay is exponential in form and the c tends to its limiting value as J tends to $J_c = \frac{1}{4}$. No modes which decay algebraically as J decreases to some value less than J_c were found; this is not surprising as the corresponding rigid-lid problem of Couette flow has no such modes either.

Overall, then, the numerical results tie in well with the analytic results of the previous section. In particular, the asymptotic forms for c at small Richardson number seem to be very good approximations indeed. Furthermore, the error in substituting a rigid boundary

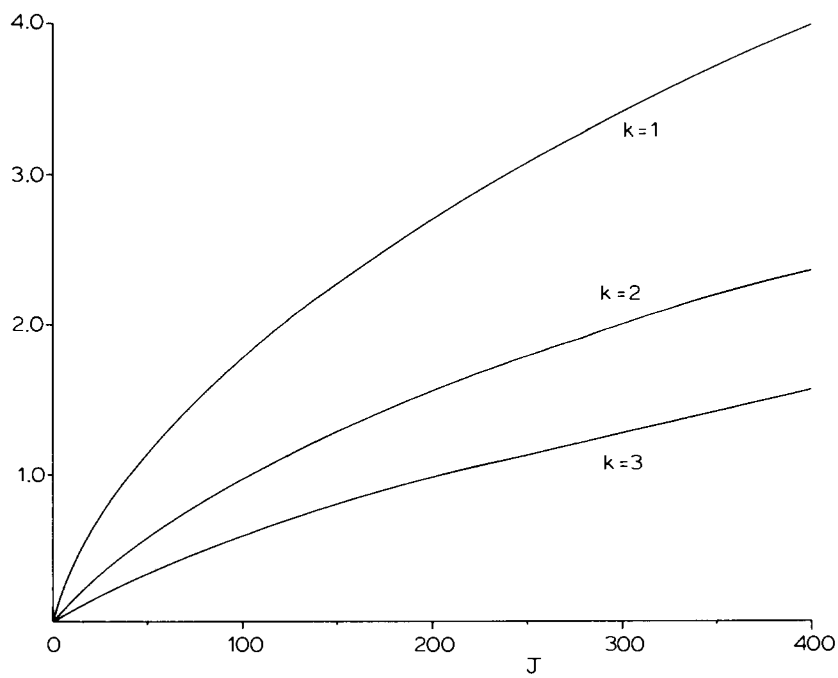

Figure 3. Wave speed c for the first three modes in the linear shear flow $\bar{u}(z) = z$, for $\alpha^2 = 10$ and $G = 10^5$.

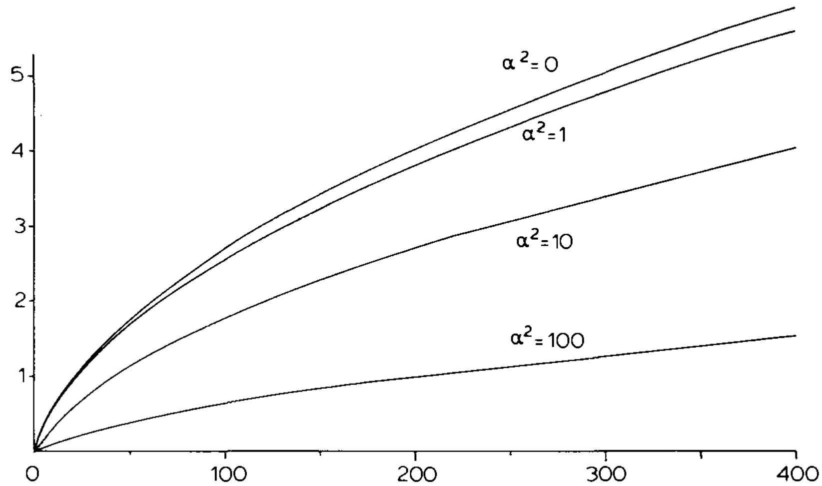

Figure 4. Wave speed c of the lowest mode in the linear shear flow with $G = 10^5$ and wave number α, $\alpha^2 = 0$, 1, 10 and 100.

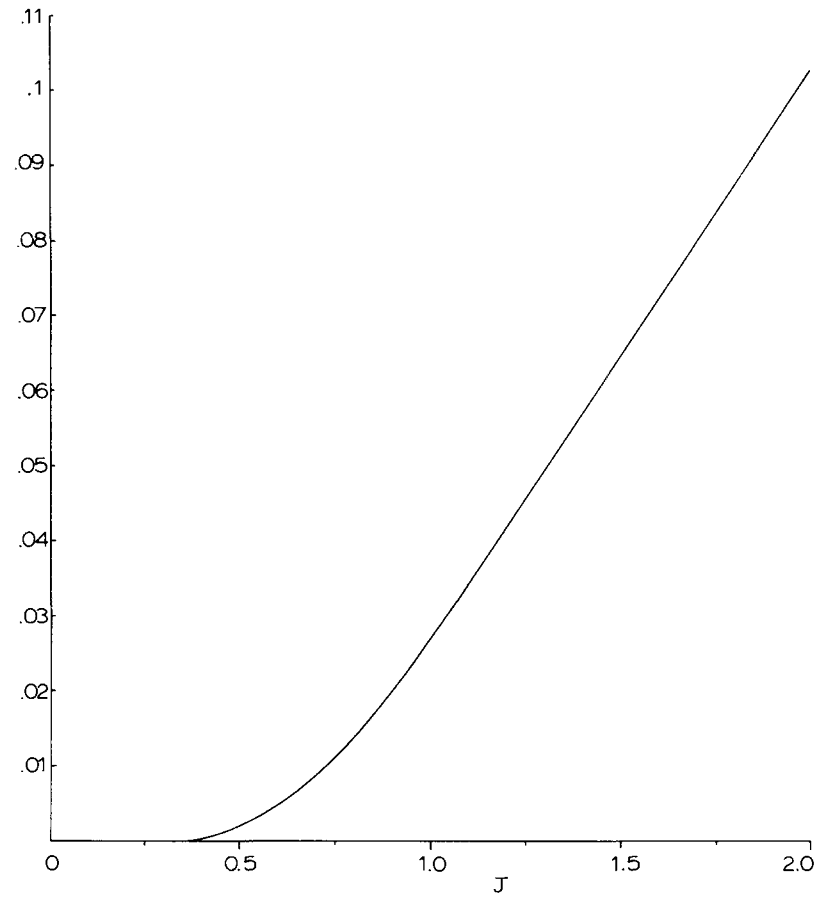

Figure 5. Wave speed c of the lowest mode in the linear shear flow, at small Richardson number, showing the exponential decay as $J \to \frac{1}{4}$; $\alpha = 0$, $G = 10^5$.

for the free surface can be seen to be small and such an approximation hence justifiable, especially in the range of parameters likely to be of interest in oceanic applications.

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